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Propagation of a relativistic particle in terms of the unitary irreducible representations of the Lorentz group

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Abstract. In a generalized Heisenberg/Schrödinger picture we use an invariant space-time transformation to describe the motion of a relativistic particle. We discuss the relation with the relativistic mechanics and find that the propagation of the particle may be defined as space-time transition between states with equal eigenvalues of the first and second Casimir operators of the Lorentz algebra. In addition we use a vector on the light-cone. A massive relativistic particle with spin 0 is considered. We also consider the nonrelativistic limit.

1 Introduction

In this paper we present a new mathematical formalism for describing the motion of a relativistic particle which is based on the principal series of the unitary irreducible representations of the Lorentz group and a generalized Heisenberg/Schrödinger picture. The principal series of the representations of the Lorentz group has already been used by many authors in the theory of elementary particles and relativistic nuclear physics (e.g., [1–7]). In our previous papers [8–10] it has been shown that these representations may be used in a generalized Heisenberg/Schrödinger picture in which either the analogue of Heisenberg states or the analogue of Schrödinger operators are independent of both time and space coordinates t, $\mathbf{x}$. For these states there must be space-time independent expansion. If at first we use the momentum representation in the expansion of the Lorentz group, then the states and operators of the Poincaré algebra can be constructed in another space-time independent representation.

In [8] the transition from the Heisenberg to the Schrödinger picture in quantum mechanics $S(t) = \exp(-itH)$ was generalized to the relativistic invariant transformation (we choose here a system of units such that $\hbar = 1$, $c = 1$)

$$S(x) := \exp[-i(tH - \mathbf{x} \cdot \mathbf{P})], \tag{1.1}$$

where H and $\mathbf{P}$ are the Hamilton and momentum operators of the particle in the generalized Schrödinger picture. Through this transformation the plane waves $\sim \exp[-ixp]$ appear in different representation and cannot be used in their original sense as the stationary states of a particle. There is no $\mathbf{x}$ representation. In this approach one must find a new method for describing the motion of the particle.

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In the present paper we will show that using the transformation $S(x)$ makes it possible to describe the motion of a massive relativistic particle in terms of the matrix elements of the Lorentz group. First we introduce in the relativistic mechanics the analogue of the operators of the Lorentz algebra in the generalized Heisenberg picture and obtain equations which in the invariant form express the motion of a particle. Then we use these equations and the unitary irreducible representations of the Lorentz group to determine the transiton amplitudes for the free relativistic particle. In particular we consider a massive particle with spin 0. In the nonrelativistic limit we use the expansion of the Galileo group.

2 Lorentz algebra in the generalized Heisenberg picture. Analogue in relativistic mechanics

In the generalized Heisenberg/Schrödinger picture the analogue of Schrödinger operators of a particle are defined as space-time independent operators in different representations. In the momentum representation ($\mathbf{p}$ = momentum, m = mass, $p_0 := \sqrt{m^2 + \mathbf{p}^2}$, s = spin) the boost and rotation generators of the Lorentz group

$$\mathbf{N} := ip_0 \nabla_{\mathbf{p}} - \frac{\mathbf{s} \times \mathbf{p}}{p_0 + m}, \qquad \mathbf{J} = -i\mathbf{p} \times \nabla_{\mathbf{p}} + \mathbf{s} := \mathbf{L}(\mathbf{p}) + \mathbf{s}. \tag{2.1}$$

can be viewed as such operators. Using the transformation $S(x)$ we obtain the operators of the Lorentz algebra in the generalized Heisenberg picture

$$\mathbf{N}(x) = S^{-1}(x)\mathbf{N}S(x) = \mathbf{N} + t\mathbf{P} - \mathbf{x}H, \tag{2.2}$$

$$\mathbf{J}(x) = S^{-1}(x)\mathbf{J}S(x) = \mathbf{J} - \mathbf{x} \times \mathbf{P}. \tag{2.3}$$

Time and space coordinates equally occur in these operators. From this point of view one can see $\mathbf{N}(x)$, $\mathbf{J}(x)$ as field operators which satisfy the equations

$$\frac{\partial N_i(x)}{\partial t} = P_i, \quad \frac{\partial N_i(x)}{\partial x_j} = -H\delta_{ij}, \quad \frac{\partial J_i(x)}{\partial x_j} = -\epsilon_{ijk}P_k, \tag{2.4}$$

and the commutation rules of the Lorentz algebra

$$[N_i(x), N_j(x)] = -\imath\epsilon_{ijk}J_k(x),$$
$$[N_i(x), J_j(x)] = \imath\epsilon_{ijk}N_k(x), \tag{2.5}$$
$$[J_i(x), J_j(x)] = \imath\epsilon_{ijk}J_k(x). \tag{2.6}$$

For the Casimir operators we have

$$C_1(x) := \mathbf{N}^2(x) - \mathbf{J}^2(x), \quad C_2(x) := \mathbf{N}(x)\cdot\mathbf{J}(x). \tag{2.7}$$

We introduce the field $\mathbf{N}(x)$, $\mathbf{J}(x)$ in the relativistic mechanics and use the same symbols. In the problem which we discuss one must find the property of the field $\mathbf{N}(x)$, $\mathbf{J}(x)$ and the invariant $C_1(x)$, $C_2(x)$ along the trajectory of a particle.

Let us write (2.2) and (2.3) in the relativistic mechanics in the form

$$\mathbf{N}(x) = \mathbf{N}(t_0, \mathbf{x}_0) + (t - t_0)\mathbf{P} - (\mathbf{x} - \mathbf{x}_0)H, \tag{2.8}$$
$$\mathbf{J}(x) = \mathbf{J}(t_0, \mathbf{x}_0) - ((\mathbf{x} - \mathbf{x}_0) - (t - t_0)\mathbf{P}/H) \times \mathbf{P}, \tag{2.9}$$

where $\mathbf{x}_0$ are the position of the particle on the time t_0. For the trajectoty ($\mathbf{x}_t = \mathbf{x}_0 + (t - t_0)\mathbf{P}/H$) we have

$$\mathbf{N} + t\mathbf{P} - \mathbf{x}_t H = \mathbf{N} + t_0\mathbf{P} - \mathbf{x}_0 H,$$
$$\mathbf{J} - \mathbf{x}_t \times \mathbf{P} = \mathbf{J} - \mathbf{x}_0 \times \mathbf{P}. \tag{2.10}$$

In these formulas the quantity $\mathbf{N}$, $\mathbf{J}$ are separated from integrals of the motion $t_0\mathbf{P} - \mathbf{x}_0 H$, $\mathbf{x}_0 \times \mathbf{P}$ because the operators $\mathbf{N}$, $\mathbf{J}$ in the generalized Heisenberg/Schrödinger picture correspond to the space-time independent quantity. For two points of the trajectoty we obtain

$$\mathbf{N}(t_1, \mathbf{x}_1) = \mathbf{N}(t_2, \mathbf{x}_2), \quad \mathbf{J}(t_1, \mathbf{x}_1) = \mathbf{J}(t_2, \mathbf{x}_2), \tag{2.11}$$

and come to the conclusion that

$$C_1(t_1, \mathbf{x}_1) = C_1(t_2, \mathbf{x}_2), \quad C_2(t_1, \mathbf{x}_1) = C_2(t_2, \mathbf{x}_2). \tag{2.12}$$

These equations represent the motion of a particle from the point $t_1, \mathbf{x}_1$ to the point $t_2, \mathbf{x}_2$ in the invariant form and may be used in the quantum version.

In connection with (2.10) we make the following remarks. The space-time parts in (2.2) and (2.3) have the structure of the four-tensor of angular momentum of a particle. If we assume contradictorally to the concept of the generalized Heisenberg/Schrödinger picture that the operators $\mathbf{N}, \mathbf{J}$ in the form

$$\mathbf{x_t} = \mathbf{N}/H + t\mathbf{P}/H, \quad \mathbf{J} = \mathbf{x_t} \times \mathbf{P} \tag{2.13}$$

correspond in the relativistic mechanics to integral of the motion then we arrive at the $\mathbf{x}$ representation and

$$\mathbf{N}(t, \mathbf{x}_t) = 0, \quad \mathbf{J}(t, \mathbf{x}_t) = 0,$$
$$C_1(t, \mathbf{x}_t) = 0, \quad C_2(t, \mathbf{x}_t) = 0. \tag{2.14}$$

In this case the zero on the right-hand side in (2.14) makes the inverse transition from the relativistic mechanics to the quantum version impossible.

3 Transition amplitudes

In the quantum version the equations (2.12) correspond to the transition $S(x_2 - x_1)$ of the particle from one state to another state with equal eigenvalues of the operators

$$C_1 = \mathbf{N}^2 - \mathbf{J}^2, \quad C_2 = \mathbf{N}\cdot\mathbf{J}. \tag{3.1}$$

For the principal series the eigenvalues of the operators C_1 and C_2 are $1 + \alpha^2 - \lambda^2$ and $\alpha\lambda$, $(0 \leq \alpha < \infty, \quad \lambda = -s, ..., s)$, respectively. The representations (α, λ) and $(-\alpha, -\lambda)$ are unitarily equivalent. In the momentum representations for a massive particle with spin zero we use the eigenfunctions of the operators C_1

$$\xi^{(0)}(\mathbf{p}, \alpha, \mathbf{n}) := \frac{1}{(2\pi)^{3/2}}[(pn)/m]^{-1+i\alpha}, \tag{3.2}$$

here $n := (\mathbf{n}, n_0)$ is a vector on the light-cone ($\mathbf{n}^2 - n_0^2 = 0$). These functions were used first in [6] in the space-time independent expansions of the Lorentz group ($n_0 = 1, \mathbf{n} := (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$)

$$\Psi^{(0)}(\mathbf{p}) = \int \alpha^2 d\alpha\, d\omega_{\mathbf{n}}\, \Psi^{(0)}(\alpha, \mathbf{n})\, \xi^{(0)}(\mathbf{p}, \alpha, \mathbf{n}), \tag{3.3}$$

$$\Psi^{(0)}(\alpha, \mathbf{n}) = \int \frac{d\mathbf{p}}{p_0}\, \Psi^{(0)}(\mathbf{p})\, \xi^{*(0)}(\mathbf{p}, \alpha, \mathbf{n}) \tag{3.4}$$

where $\Psi^{(0)}(\mathbf{p})$ and $\Psi^{(0)}(\alpha, \mathbf{n})$ are the state functions of the particle with spin zero in $\mathbf{p}$ and in the $\alpha, \mathbf{n}$ representation. The Hamilton operator $H^{(0)}(\alpha, \mathbf{n})$ and momentum operators $\mathbf{P}^{(0)}(\alpha, \mathbf{n})$ were constructed in [3]. The operators $\mathbf{N}$ in the $\alpha, \mathbf{n}$ representation have the form ([8–10])

$$\mathbf{N} := \alpha\mathbf{n} + (\mathbf{n} \times \mathbf{L} - \mathbf{L} \times \mathbf{n})/2, \quad \mathbf{L} := \mathbf{L}(\mathbf{n}). \tag{3.5}$$

For the particle with spin s

$$\mathbf{N} : = \alpha\mathbf{n} + (\mathbf{n} \times \mathbf{J} - \mathbf{J} \times \mathbf{n})/2 \quad \mathbf{J} := \mathbf{L}(\mathbf{n}) + \mathbf{s}, \tag{3.6}$$
$$C_1 = 1 + \alpha^2 - (\mathbf{s}\cdot\mathbf{n})^2, \quad C_2 = \alpha\mathbf{s}\cdot\mathbf{n}, \tag{3.7}$$
$$[C_1, \mathbf{n}] = 0, \quad [C_2, \mathbf{n}] = 0. \tag{3.8}$$

and as a complete set of commuting operators one can select the invariants C_1, C_2 and the vector $\mathbf{n}$.

In the relativistic mechanics we must find the property of the vector $\mathbf{n}$ along the trajectory of the particle. In accordance with formulas (3.5) in the relativistic mechanics the quantity $\mathbf{N}$, the field $\mathbf{N}(x)$ and the invariant $C_1(x)$ can be expressed in the form

$$\mathbf{N} := \alpha\mathbf{n} + \mathbf{n} \times \mathbf{L}(\mathbf{n}), \quad C_1 = \alpha^2, \tag{3.9}$$
$$\mathbf{N}(x) := \alpha(x)\mathbf{n}(x) + \mathbf{n}(x) \times \mathbf{L}(x),$$
$$C_1(x) = \alpha^2(x). \tag{3.10}$$

Using (2.11) and (2.12) we obtain

$$\alpha^2(t_1, \mathbf{x}_1) = \alpha^2(t_2, \mathbf{x}_2), \quad \mathbf{n}(t_1, \mathbf{x}_1) = \mathbf{n}(t_2, \mathbf{x}_2). \tag{3.11}$$

Let $|\alpha, \lambda, \mathbf{n}\rangle$ be the states with a well-defined value of the operator C_1, C_2 and the vector $\mathbf{n}$. Then in accordance with (2.12) and (3.11) for the transition amplitude for a massive relativistic particle we have the expression in terms of the matrix elements of the unitary irreducible representations of the Lorentz group

$$K(x_2; x_1, \alpha, \lambda, n) \\ := \langle \alpha, \lambda, \mathbf{n}|S(x_2 - x_1)|\mathbf{n}', \lambda', \alpha'\rangle_{\alpha,\lambda,\mathbf{n}=\alpha',\lambda',\mathbf{n}'}. \tag{3.12}$$

For example for a particle with spin zero

$$K(x_2; x_1, n) = \left(\int \frac{d\mathbf{p}}{p_0}\, \xi^{*(0)}(\mathbf{p}, \alpha, \mathbf{n})\, S(x_2 - x_1) \times \xi^0(\mathbf{p}, \alpha', \mathbf{n}') \right)_{\alpha,\lambda,\mathbf{n}=\alpha',\lambda',\mathbf{n}'} = \frac{1}{(2\pi)^3} \int \frac{d\mathbf{p}}{p_0} \frac{\exp -i[(x_2 - x_1)p]}{[(pn)/m]^2}. \tag{3.13}$$

This transition amplitude contain the vector of the light-cone n. Applying the operator $[i(n_0\partial_t + \mathbf{n}\nabla_{\mathbf{x}})/m]^2$ we have the relation to the Feynman propagator $\triangle^+(x)$ of the free Klein-Gordon equation

$$\triangle^+(x) = \frac{-i}{2}[i(n_0\partial_t + \mathbf{n}\nabla_{\mathbf{x}})/m]^2 K(x, n), \tag{3.14}$$

where

$$\triangle^+(x) = \frac{-i}{(2\pi)^3} \int \frac{d\mathbf{p}}{2p_0} \exp -i[px]. \tag{3.15}$$

In the nonrelativistic limit in the momentum representation $\mathbf{N} \to im\nabla_{\mathbf{p}} := \mathbf{q}$ and

$$\xi^{(0)}(\mathbf{p}, \alpha, k) \;\to\; \Psi(\mathbf{p}, \alpha\mathbf{n}) := \frac{1}{(2\pi)^{3/2}} \exp[-i(\alpha\mathbf{n}) \cdot \mathbf{p}/m]. \tag{3.16}$$

The functions $\Psi(\mathbf{p}, \alpha\mathbf{n})$ are the eigenfunctions of the operators $\mathbf{q}$ and $\mathbf{q}^2$. In [10]) was remarked that in the expansion of the Galileo group

$$\Psi(\alpha\mathbf{n}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{p}\Psi(\mathbf{p}) \exp(i\alpha\mathbf{n} \cdot \mathbf{p}/m), \tag{3.17}$$

where $\Psi(\mathbf{p})$ and $\Psi(\alpha\mathbf{n})$ are the states of the particle in $\mathbf{p}$ and in the $\alpha, \mathbf{n}$ representation the kernel $\exp(i\alpha\mathbf{n} \cdot \mathbf{p}/m)$ can be replaced by plane waves $\exp(i\mathbf{x} \cdot \mathbf{p})$ and in such a form the $\mathbf{x}$ representation in the nonrelativistic limit can be constructed. It is well known that the transition amplitude may be written in this case as $\langle \mathbf{x}_2|S(t_2 - t_1)|\mathbf{x}_1\rangle$. In the relativistic region this method cannot be used. The functions which realize the unitary irreducible space-time independent representations of the Lorentz group and of the Galileo group have different forms.

In the framework of the generalized Heisenberg/Schrödinger picture for describing the motion of a particle in the nonrelativistic limit we can use the same method as in the relativistic case. The operators

$$\mathbf{q}(t, \mathbf{x}) = \mathbf{q} + t\mathbf{p} - \mathbf{x}m, \\ \mathbf{q}(t, \mathbf{x}) = \mathbf{q}(t_0, \mathbf{x}_0) + (t - t_0)\mathbf{p} - (\mathbf{x} - \mathbf{x}_0)m, \tag{3.18}$$

just as $\mathbf{N}(x)$ can be viewed as field operators with the property

$$\frac{\partial q_i(t, \mathbf{x})/m}{\partial t} = P_i/m, \quad \frac{\partial q_i(t, \mathbf{x})/m}{\partial x_j} = -\delta_{ij}. \tag{3.19}$$

In order to find the transition amplitude we must calculate the matrix element ($H = \mathbf{p}^2/2m$)

$$K(x_2; x_1, \alpha\mathbf{n}) := \langle \alpha\mathbf{n}|S(t_2 - t_1, \mathbf{x}_2 - \mathbf{x}_1)|\mathbf{n}'\alpha'\rangle_{\alpha\mathbf{n}=\alpha'\mathbf{n}'} \tag{3.20}$$

which correspond to the transition $S(t_2 - t_1, \mathbf{x}_2 - \mathbf{x}_1)$ of the particle between states with equal eigenvalues of the operator $\mathbf{q}$ and $\mathbf{q}^2$. Using the functions $\Psi(\mathbf{p}, \alpha\mathbf{n})$ we obtain expression

$$K(x_2; x_1, \alpha\mathbf{n}) = \left(\int d\mathbf{p}\Psi^*(\mathbf{p}, \alpha\mathbf{n}) S(t_2 - t_1, \mathbf{x}_2 - \mathbf{x}_1) \times \Psi(\mathbf{p}, \alpha'\mathbf{n}') \right)_{\alpha\mathbf{n}=\alpha'\mathbf{n}'} \tag{3.21}$$

$$= \left[\frac{m}{2\pi\imath(t - t_0)} \right]^{3/2} \exp \frac{\imath m(\mathbf{x}_2 - \mathbf{x}_1)^2}{2(t_2 - t_1)}.$$

which agree with the transition amplitude in quantum mechanics.

4 Conclusion

We have shown that the propagation of a relativistic particle may be described in terms of the transformation $S(x) = \exp[-i(tH - \mathbf{x} \cdot \mathbf{P})]$ and the matrix elements of the unitary irreducible representations of the Lorentz group. We have considered the operators of the Lorentz algebra in the generalized Heisenberg picture as field operators and found that the analogue of the space-time independent operators in the relativistic mechanics must be separated from the integrals of motion. In this case the conversion to the quantum version can take place. The transition amplitude for a particle with spin zero contain a vector of the light-cone which appears in the expansions of the Lorentz group. Finally we have shown that in the nonrelativistic limit the transition amplitude may be also expressed in terms of the transformation $S(x)$.

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